Topic Study Group 23

Visualization in the Teaching and Learning of Mathematics

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1. Introduction and Aims of the TSG

In mathematics education research, visualization is generally referred to as the product and the process of creating, using, interpreting, and reflecting on visual information. It plays an important role in mathematical thinking and in most branches of mathematics: there is general consensus in the mathematics education community that visualization is a vital component of conceptual understanding, reasoning, problem solving and proving.

The aim of the TSG-23 was to interrogate the significance for research in understanding the role of visualization processes in the teaching and learning of mathematics at all school levels. Specifically, it was the aim of TSG-23 to not only showcase this research in a global context, but also to start thinking about and considering possible visualization research trajectories and frameworks that could support scholars with articulating their own visualization research agendas.

In the call for papers we proposed some subthemes, which highlighted the close connection of visualization with different aspects involved in mathematics learning and teaching, such as:

- *Visualization as a cognitive process*, including visualization and reasoning, justification, argumentation, imagination, and difficulties with visualization.
- *Visualization as a mathematical construct*, including visualization and mathematizing, visualization and generalizing, visualization as a mathematical proof.
- *Visualization and new technologies*, including technologies such as interactive dynamic software, 3-D printing, augmented reality, virtual reality and other digital media.
- *Visualization and neurological functioning*, including research into neurological activities in the brain associated with visualization processes and

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their implications to mathematical thinking and teaching-learning processes

- *Visualization and language*, including interrogating the relationships between visualization, signs and language(s), including embodied aspects such as gestures and bodily actions.
- *Visualization in school practice and in teacher education*, including research into the explicit inclusion of visualization in school curriculum, practice and assessment, and interrogating the development of visualization skills in teacher education programs.
- *Visualization as a social process*, including research about the negotiations on visualization in classrooms as well as diversity aspects across various cultural contexts.
- *Visualization and research methodology*, including asking questions about visualization research design and methodological approaches that foster visualization.
- *Visualization and theory*, including research into possible overarching theoretical frameworks that could frame and orient visualization research such as embodied cognition theories, learning theories, socio-cultural theories.

This list was not meant to be exhaustive, and we encouraged contributions based on empirical research, including ongoing studies, as well as theoretical elaborations and reflections on a theme.

2. Submissions

We received 29 submissions of papers and 13 submissions of posters from 16 countries (South America and North America, Asia, Europe and Africa), which shows a very encouraging cultural diversity. Papers underwent a peer review process that involved two submitting authors and one team member for each paper submission, and two team members for each poster submission. The review process was guided by criteria that included innovation, theory and methodology, coherence, interest for an international audience and clarity. Paper proposals could be accepted as either long or short papers. After the review process was completed, and the conference finally took place, 13 long papers and 9 short papers were presented and discussed during the conference sessions.

3. Sessions

During the conference, we met in a blended modality (in presence and at distance) in four sessions, for a total of six hours. Onsite facilitators in Shanghai helped us to manage a reliable connection between onsite and online participants.

As TSG-23 we wanted to provide sufficient time for scientific exchange and discussion. As we had not enough time to discuss and interact with the presenters during our official sessions, we met in additional interactive sessions during dinner times (Shanghai times) in order to share moments of scientific discussion and exchange pertaining to the specific presentations. These meetings were realized only at distance

and made use of the parallel rooms facilities. Throughout the four days we attempted to facilitate dialogue between participants in order to give constructive feedback to the presented studies, identify possible research themes and opportunities that emerged from the presentations. We found that the discussion time proved very fruitful in facilitating networking opportunities and sustaining the interest momentum that was generated during the presentations. At the end of the last session, we hosted a 50-minute whole group reflection session to consider some research implications that arose.

4. Paper Presentation and Emerging Themes

The long papers (LO) were allocated 15 minutes for presentation, and short papers (SO) were allocated 10 minutes. Papers were briefly discussed in the official sessions, and then discussed in more depth in the parallel rooms during the additional sessions.

Our programme was organized around eight themes or clusters which characterized the submissions. Below we present each theme as they appeared in the programme and list the related papers in Tab.1 (on the next page).

- *Theme 1: Visualization and problem-solving.* This theme related specifically to research that looked at how visualisation process and mathematical problem-solving articulated with each other.
- *Theme 2: Classroom interaction.* This theme interrogated how issues of visualization played out in particular classroom interaction contexts.
- *Theme 3: Visualization and teaching*. The focus of this theme was how selected teachers used visualization tools and media (including dynamic geometry software) to teach mathematics.
- *Theme 4: Different kinds of representations, different technologies.* In this theme the presenters looked at how different representations and diagrams could be meaningfully utilized in the mathematics classroom.
- *Theme 5: Diagrams and mathematics visualization.* Here the specific focus was on how mathematical visualization related to mathematical representations.
- *Theme 6: Math, visualization and other disciplines.* In this theme researchers engaged with mathematical visualization processes using means from other disciplines.
- *Theme 7: Visualization and (latest) technologies.* Here researchers considered how different digital resources were used either as a research tool or means to engage with mathematics.
- *Theme 8: Educational materials.* In this theme the presenters considered how specific educational materials, in the context of visualization processes, could be used in the mathematics classroom.

Tab. 1. List of papers presented

Paper and author(s)
Session 1
Theme 1: Visualization and problem-solving
[1] Imaging and visualizing in geometry: Explorations by mathematics university students. ***Ferdinando Azarello****, Cristina Sabena, and Carlotta Soldano* (Italy). (LO)
[2] Visualization as an embodied problem-solving process. ***Beata Dongwi*** *and Marc Schäfer* (South Africa). (LO)
[3] Characterizing visualization and spatial analytic reasoning for solving isometry problems. ***Leah Michelle Frazee*** *and Michael Battista* (USA). (LO)
[4] The role of visualization towards student's mathematical abstraction and representation: The case of probability. ***Dennis Lee Jarvis Baring Ybanez*** *and Catherine Vistro-Yu* (Philippines). (LO)
Theme 2: Classroom interaction
[5] The use of gestures and language as co-existing visualization teaching tools in multilingual classes. ***Clemence Chikiwa*** *and Marc Schäfer* (South Africa). (LO)
[6] On objects and visualizations — An interactionistic perspective. ***Marei Fetzer*** (Germany). (LO)
Session 2
Theme 3: Visualization and teaching
[7] How teachers scaffold students in visualizing diagram for understanding geometric problem solving. ***Hui-Yu Hsu*** (Chinese Taipei). (LO)
[8] Preservice and Inservice teachers' mathematics visualization skills. ***Vimolan Mudaly*** (South Africa). (LO)
[9] Dynamic visual instructions by GeoGebra for introducing Takada's theorem on pentagons. ***Hirotshi Furutsu****, Yukiko Ishii, Hisashi Kato, Yusuke Washio, and Noriko Hirata-Kohno* (Japan). (SO)
[10] High school mathematics inquiry teaching based on GeoGebra visualization environment. ***Wei Wang*** *and Xue Huang* (China). (SO)
Theme 4: Different kinds of representations, different technologies
[11] The development of 3D representations using physical manipulatives, technology-aided design and 2D drawings. ***Jill A. Cochran*** (USA). (LO)
[12] The social construction of knowledge in a new pedagogical setting: The same activity presented as three different interactive diagrams. ***Elena Navtaliev*** (Israel). (LO)
Session 3
Theme 5: Diagrams and mathematics visualization
[13] Visualization as vision, imagination and intuition: reflections on graduate students struggling with a visual conjecturing problem. ***Francesco Beccuti*** (Italy). (LO)
[14] Mapping diagrams: Function visualization of real and complex analysis and matrix algebra. ***Martin Flashman*** (USA). (SO)
[15] Interactive visualizations of topics in engineering mathematics. ***Antti Rasila*** (China). (SO)
Theme 6: Math, visualization and other disciplines
[16] Drawing (on) diagrams: Typicality of geometric shapes in concept image elicitation for secondary students. ***Santanu Dutta****, Charudatta Sharad Navare, and Harita Raval* (India). (SO)
[17] Research on visualization in mathematics learning based on mathematical drama performance or by video. ***Yan Li****, Pan Liu, and Xinyu Liu* (China). (SO)
Session 4
Theme 7: Visualization and (latest) technologie
[18] Some like it social: Looking into the interplay between math and internet memes. ***Giulia Bini*** *and Ornella Robutti* (Italy). (LO)
[19] Children's ambiguous interpretation of visualizations – eye tracking as a diagnostic tool for division concepts. ***Daniela Götze*** (Germany). (LO)
[20] A review of the application cases of augmented reality (ar) in mathematics education. ***Luona Wang*** (China). (SO)
Theme 8: Educational materials
[21] Using geometric intuition in the domain of number and algebra: From textbook designers' perspective. ***Jiling Gu*** *and Fei Zhang* (China). (SO)
[22] Methodology visual experience based mathematics education 2019. ***Janos Szasz Saxon*** *and Zsuzsa Dardai* (Hungary). (SO)

5. Areas for Future Research

In our last session we consolidated our deliberations by identifying possible visualisation avenues for further research that emerged. These were:

- Visualisation as a cognitive process. This includes visualisation and reasoning, visualisation and imagination, difficulties with visualisation.
- Visualisation and mathematising.
- Visualisation and new technologies, such as interactive dynamic software, augmented reality and other digital media.
- Visualisation and language, specifically the relationship between visualisation, signs, language(s), including embodied aspects such as gestures and bodily actions.
- Visualisation as a social process, including negotiations on visualisation in classrooms as well as diversity aspects across various cultural contexts.
- Visualisation and theory, which includes researching possible overarching theoretical frameworks that could frame and orient visualisation research such as embodied cognition theories, learning theories and socio-cultural theories. Do we have a visualisation theory?